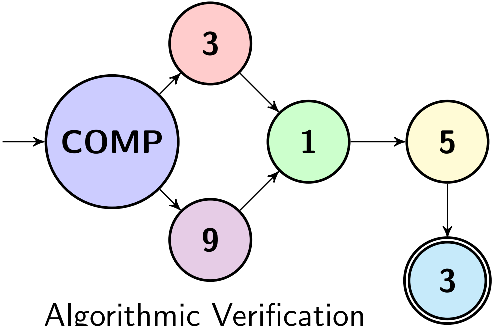

Algorithmic Verification

Temporal Logics

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Reachability

Define $\text{Reach}(A, q) \subseteq Q$ as the set of states reachable in A from q.

Define $\text{Reach}(A) \equiv \text{Reach}(A, q_0)$.

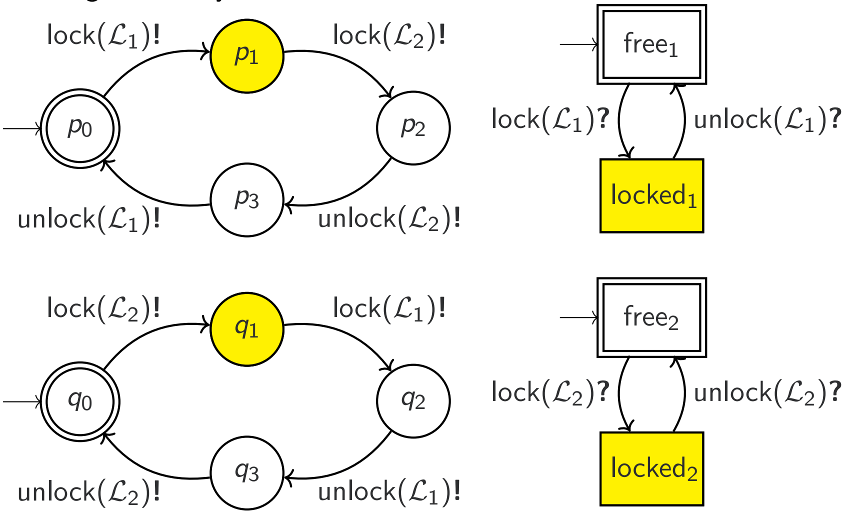
Exercise

Describe the algorithm for computing $\text{Reach}(A)$.

Deadlock or a stuck state is a state $q \in Q$ which has no outgoing transitions i.e $\forall a.\ \delta(q, a) = \emptyset$.

Deadlock Example

Assuming unicast synchronisation:



Exercise: What is an algorithm to detect deadlock?

Safety Properties

A safety property is an assertion that bad things do not happen. In other words, given some set of states $\mathsf{Bad} \subseteq Q$, we want to check that:

$$\mathsf{Bad} \cap \mathsf{Reach}(A) = \emptyset$$

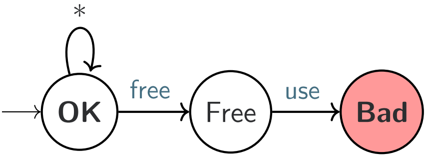
Exercise

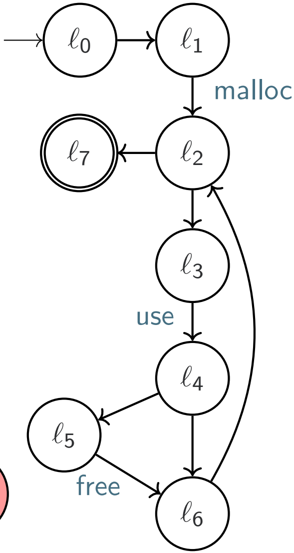
Give an algorithm to check a safety property.

Observations

Is use after free a safety property?

```
void foo() {
   int x, a;
   int *p = malloc(sizeof(int));
   for (x = 10; x > 0; x--) {
      a = *p;
      if (x <= 1) {
         free(p);
      }
   }
}
```

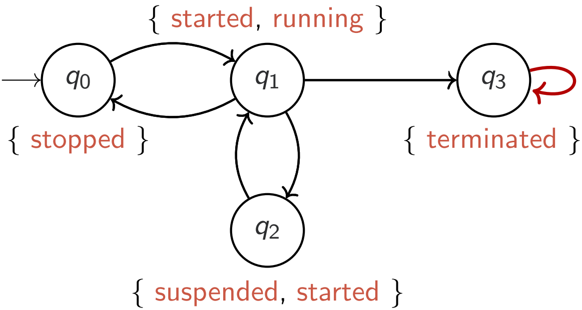




Kripke Structures

Definition

A *labelled automaton* is a FA $(Q, q_0, \Sigma, \delta, F, L)$ with an additional labelling function $L : Q \to 2^{\mathcal{P}}$, where $\mathcal{P}$ is our atomic propositions. A *Kripke structure* is a type of labelled automaton where $|\Sigma| = 1$, $F = Q$. Equivalently, we don't have a notion of actions or final states, and $\delta : Q \to 2^{Q}$. We also require that for any q, $\delta(q) \neq \emptyset$.



Traces

Definition

A *trace*, also called a *behaviour*, is the sequence of labels corresponding to a run. For Kripke structures it is necessarily infinite in length.

Define $\mathrm{Traces}(A)$ to be all possible infinite traces from q_0 in A.

Definition

A linear time *property* is a set of traces, i.e. a subset of $(2^{\mathcal{P}})^{\omega}$. We say a Kripke structure A *satisfies* a property P iff:

$$\mathrm{Traces}(A) \subseteq P$$

LTL

Linear temporal logic (LTL) is a *logic* designed to describe linear time properties.

Linear temporal logic syntax

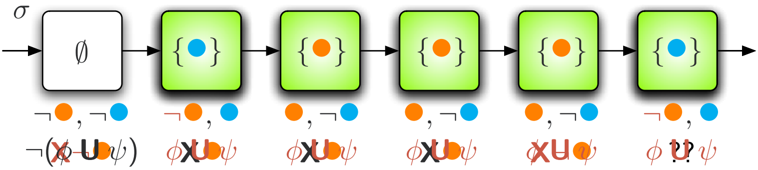
We have normal propositional operators:

- $p \in \mathcal{P}$ is an LTL formula.
- If φ, ψ are LTL formulae, then $\varphi \wedge \psi$ is an LTL formula.
- If φ is an LTL formula, $\neg\varphi$ is an LTL formula.

We also have modal or temporal operators:

- If φ is an LTL formula, then $\mathbf{X}\ \varphi$ is an LTL formula.
- If φ, ψ are LTL formulae, then φ **UNTIL** ψ is an LTL formula.

LTL Semantics in Pictures



LTL Semantics

Let $\sigma = \sigma_0\sigma_1\sigma_2\sigma_3\sigma_4\sigma_5\ldots$ be a trace. Then define notation:

- $\sigma|_0 = \sigma$
- $\sigma|_1 = \sigma_1\sigma_2\sigma_3\sigma_4\sigma_5\ldots$
- $\sigma|_{n+1} = (\sigma|_1)|_n$

Semantics

The models of LTL are traces. For atomic propositions, we just look at the first state:

$$
\begin{array}{lcl}
\sigma \models p & \Leftrightarrow & p \in \sigma_0 \\
\sigma \models \varphi \wedge \psi & \Leftrightarrow & \sigma \models \varphi \text{ and } \sigma \models \psi \\
\sigma \models \neg\varphi & \Leftrightarrow & \sigma \not\models \varphi \\
\sigma \models \mathbf{X}\ \varphi & \Leftrightarrow & \sigma|_1 \models \varphi \\
\sigma \models \varphi\ \mathbf{UNTIL}\ \psi & \Leftrightarrow & \text{There exists an } i \text{ such that } \sigma|_i \models \psi \\
 & & \text{and for all } j < i,\ \sigma|_j \models \varphi
\end{array}
$$

We say $A \models \varphi$ iff $\forall\sigma \in \text{Traces}(A).\ \sigma \models \varphi$.

Derived Operators

The operator $\mathbf{F}\ \varphi$ ("finally" or "eventually") says that φ will be true at some point.

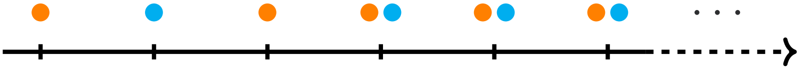
The operator $\mathbf{G}\ \varphi$ ("globally" or "always") says that φ is always true.

Exercise

- Give the semantics of **F** and **G**.
- Define **F** and **G** in terms of other operators.

More Exercises

Let ρ be this trace:



More Derived Operators

- Define "Infinitely Often" in LTL.
- Define "Almost Globally" in LTL (always true from some point onwards).

$\rho \models {\color{orange}\bullet}$?

$\rho \models {\color{cyan}\bullet}$?

$\rho \models \mathbf{X}\ {\color{cyan}\bullet}$?

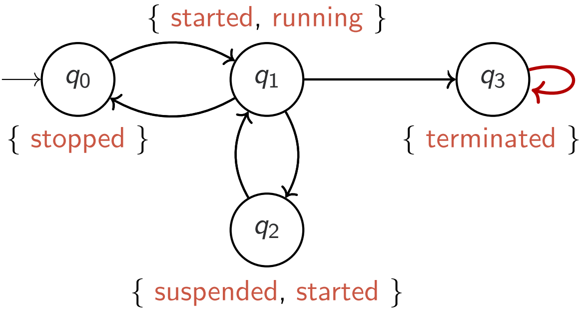
$\rho \models \mathbf{F}\ {\color{orange}\bullet}$?

$\rho|_3 \models \mathbf{F}\ ({\color{orange}\bullet} \wedge \neg {\color{cyan}\bullet})$?

$\rho \models \mathbf{FG}\ ({\color{orange}\bullet} \wedge {\color{cyan}\bullet})$?

$\rho \models \mathbf{G}\ ({\color{orange}\bullet}\ \mathbf{UNTIL}\ {\color{cyan}\bullet})$?

Possible Futures



We can see that it is always possible for a run to move to the terminated state. How do we express this in LTL? **We can't!** — it is a *branching time* property.

Branching Time

Definition

The *computation tree* of a Kripke structure A, written $\mathrm{Tree}(A)$, is an infinite tree of Kripke structure states, where q_0 is the root and a state q' is a child of q if $q' \in \delta(q)$.
A *path* $t_1 t_2 t_3 \ldots$ is a (infinite) sequence of computation trees such that t_{n+1} is the child of t_n. Define $\mathrm{Paths}(t)$ to be the set of all paths starting at t.

Exercise

Draw the CT for the process example.

CTL* Syntax

Definition

We define two types of formulae, state formulae and path formulae, named based on their models.

A state formula (SF) is defined as follows:

- All $p \in \mathcal{P}$ are SFs.
- Given SFs P and Q, $\neg P$ is a SF and $P \wedge Q$ is a SF.
- Given a PF φ, $\mathbf{E}\varphi$ and $\mathbf{A}\varphi$ are SFs.

A path formula (PF) is defined much like LTL:

- If P is a SF, then P is a PF.
- Given PFs φ and ψ, $\neg\varphi$ is a PF and $\varphi \wedge \psi$ is a PF.
- Given a PF φ then $\mathbf{X}\varphi$ is a PF.
- Given PFs φ and ψ, φ **UNTIL** ψ is a PF.

Initially, we start with state formulae (SFs).

CTL* Semantics

State Semantics

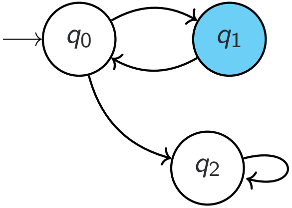
$$
\begin{array}{lcl}
t \models p & \Leftrightarrow & p \in L(t_{\text{root}}) \\
t \models P \wedge Q & \Leftrightarrow & t \models P \text{ and } t \models Q \\
t \models \neg P & \Leftrightarrow & t \not\models P \\
t \models \mathbf{E}\ \varphi & \Leftrightarrow & \exists \rho \in \text{Paths}(t).\ \rho \models \varphi \\
t \models \mathbf{A}\ \varphi & \Leftrightarrow & \forall \rho \in \text{Paths}(t).\ \rho \models \varphi
\end{array}
$$

Path Semantics

$$
\begin{array}{lcl}
\rho \models P & \Leftrightarrow & \rho_0 \models P \\
\rho \models \varphi \wedge \psi & \Leftrightarrow & \rho \models \varphi \text{ and } \rho \models \psi \\
\rho \models \neg \varphi & \Leftrightarrow & \rho \not\models \varphi \\
\rho \models \mathbf{X}\ \varphi & \Leftrightarrow & \rho|_1 \models \varphi \\
\rho \models \varphi\ \mathbf{UNTIL}\ \psi & \Leftrightarrow & \text{There exists an } i \text{ such that } \rho|_i \models \psi \\
 & & \text{and for all } j < i,\ \rho|_j \models \varphi
\end{array}
$$

CTL* Examples

We say a Kripke structure A satisfies a CTL* property P, that is,
$A \models P$ iff $\mathrm{Tree}(A) \models P$
Given this automaton A:



- $A \models \mathbf{E}\ \mathbf{G}\ \mathbf{F}$?
- $A \models \mathbf{A}\ \mathbf{G}\ \mathbf{F}$?
- $A \models \mathbf{A}\ \mathbf{F}$?
- $A \models \mathbf{A}\ \mathbf{E}\ \mathbf{F}$?

Simplifying

CTL* is very expressive but very complicated.

It's also extremely hard to model check, which we'll get to later.

CTL* to CTL

Keep state formulae the same:

- All $p \in \mathcal{P}$ are SFs.
- Given SFs P and Q, $\neg P$ is a SF and $P \wedge Q$ is a SF.
- Given a PF φ, $\mathbf{E}\varphi$ and $\mathbf{A}\varphi$ are SFs.

But we force path formulae to go straight back to state formulae immediately with a temporal operator:

- Given a SF P then $\mathbf{X}P$ is a PF.
- Given SFs P and Q, P **UNTIL** Q is a PF.

Examples

Which of the following CTL* formulae are CTL formulae?

- a **UNTIL** (b **UNTIL** c)
- **A** (a **UNTIL** c)
- **X X** a
- **X A** a
- **A** (a **UNTIL** (b **UNTIL** c))
- **A E** (a **UNTIL** b)
- **E X** a
- **X E** a

Non-mutual CTL Syntax

Simpler CTL Syntax

A CTL formula is defined as follows:

- All $p \in \mathcal{P}$ are formulae.
- Given formulae P and Q, $\neg P$ is a formula and $P \wedge Q$ is a formula.
- Given a formula P, $\mathbf{EX}\ P$ and $\mathbf{AX}\ P$ are formulae.
- Given formulae P and Q, $\mathbf{E}(P\ \mathbf{UNTIL}\ Q)$ and $\mathbf{A}(P\ \mathbf{UNTIL}\ Q)$ are formulae.

Simpler CTL Semantics

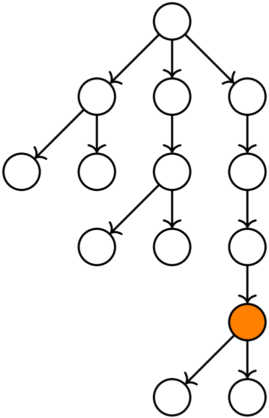
Semantics are as with CTL*, but can be defined more directly:

Semantics

$$
\begin{array}{lcl}
t \models p & \Leftrightarrow & p \in L(t_{\mathrm{root}}) \\
t \models P \wedge Q & \Leftrightarrow & t \models P \text{ and } t \models Q \\
t \models \neg P & \Leftrightarrow & t \not\models P \\
t \models \mathbf{EX}\ P & \Leftrightarrow & \exists \rho \in \mathrm{Paths}(t).\ \rho_1 \models P \\
t \models \mathbf{AX}\ P & \Leftrightarrow & \forall \rho \in \mathrm{Paths}(t).\ \rho_1 \models P \\
t \models \mathbf{A}(P\ \mathbf{UNTIL}\ Q) & \Leftrightarrow & \forall \rho \in \mathrm{Paths}(t)\text{, there } \exists \text{ an } i \text{ such that:} \\
 & & \quad \rho_i \models Q \text{ and } \forall j < i.\ \rho_j \models P \\
t \models \mathbf{E}(P\ \mathbf{UNTIL}\ Q) & \Leftrightarrow & \exists \rho \in \mathrm{Paths}(t) \text{ and an } i \text{ such that:} \\
 & & \quad \rho_i \models Q \text{ and } \forall j < i.\ \rho_j \models P
\end{array}
$$

Derived Operators

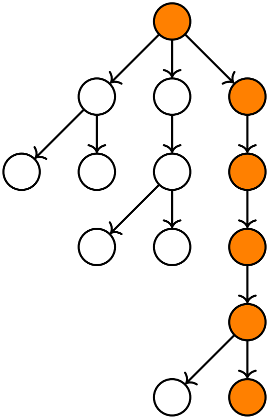
Define **EF**●:



E(True **UNTIL** ●)

Derived Operators

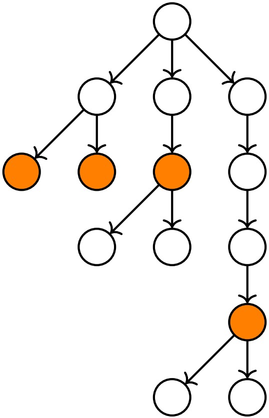
Define **EG**●:



$\neg$**A**(True **UNTIL** $\neg$●)

Derived Operators

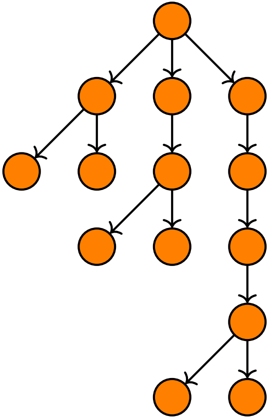
Define **AF**●:



A(True **UNTIL** ●)

Derived Operators

Define **AG**●:



$\neg$**EF** $\neg$●

Bibliography

- Huth/Ryan: Logic in Computer Science, Section 3.2 and 3.4
- Bayer/Katoen: Principles of Model Checking Sections 5.1 and 6.2